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FAST TRACK COMMUNICATION

Nonlinear dynamics of quantum systems and soliton theory

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Abstract
We show that spacetime evolution of one-dimensional fermionic systems is described by nonlinear equations of soliton theory. We identify a spacetime dependence of a matrix element of fermionic systems related to the *orthogonality catastrophe* or boundary states with the τ-function of the modified KP-hierarchy. The established relation allows us to apply the apparatus of soliton theory to the study of nonlinear aspects of quantum dynamics. We also describe a *bosonization in momentum space*—a representation of a fermion operator by a Bose field in the presence of a boundary state.

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1. Introduction

In the seminal paper by Date, Jimbo, Kashiwara and Miwa [1] Sato's approach [2] to integrable hierarchies of soliton equations has been formulated in terms of auxiliary fermions. In this formulation the hidden symmetries of soliton equations become explicit symmetries of free fermions, the modes of which are labelled by a one-dimensional parameter with respect to the algebra of fermionic bilinear forms—$gl(\infty)$. It was shown that the τ-function of soliton theory can be expressed as a certain matrix element of 1D fermions. The paper [1] and subsequent works of the Kyoto school (see [3] and references therein) laid out a mathematical foundation of *bosonization*—a commonly used representation of fermionic modes in terms of a Bose field. These works established a deep relation between quantum physics and classical nonlinear equations.

In this paper we adopt this theory to a particularly interesting area of electronic physics—dynamics of fermionic coherent states propagating in real space and time generated by the Galilean Hamiltonian

$$H = \sum_p \frac{p^2}{2M} \psi_p^\dagger \psi_p. \tag{1}$$

We will show that the real-time evolution of important objects of electronic physics also obeys classical nonlinear soliton equations. Therefore the powerful apparatus of soliton theory can be used to study complex aspects of non-equilibrium physics of effectively one-dimensional quantum systems. We will report some applications of the results of this paper to nonlinear electronic transport elsewhere. Here we focus on the formal connection of soliton theory and quantum dynamics. In this course we encounter an alternative *bosonization* with respect to the boundary state, but now in momentum space.

Before formulating our main results, we introduce notations and give a brief review of some important facts on a connection between solutions of soliton equations and fermionic matrix elements. The literature on the subject is enormous; we restrict the reference list to the original papers [1, 3].

1.1. Bilinear identity

Consider a Fermi system on a unit circle or, equivalently, 1D fermions subject to periodic boundary conditions. We let ψ_p be a fermionic mode of the fermionic field $\psi(x) = \sum_p \mathrm{e}^{\mathrm{i}px}\psi_p$, labelled by integer momenta, p. The conjugated field is given by $\psi^\dagger(x) = \sum_p \mathrm{e}^{-\mathrm{i}px}\psi_p^\dagger$ with basic (anti-) commutation relations $\{\psi_p, \psi_q\} = \left\{\psi_p^\dagger, \psi_q^\dagger\right\} = 0$ and $\left\{\psi_p, \psi_q^\dagger\right\} = \delta_{p,q}$. The ground state of the system is a filled Fermi sea such that all modes in the interval between the Fermi points are filled. We will be interested only in one, say, the right chiral sector where all essential modes are concentrated around the Fermi point $+p_F$. It is convenient to count modes from this Fermi point and change $p \to p - p_F$. Then the vacuum is the state with all non-positive momenta filled by fermions. It is defined as the state $|0\rangle$ such that

$$\psi_p|0\rangle = 0, \qquad \text{if} \quad p > 0, \qquad \psi_p^\dagger|0\rangle = 0, \qquad \text{if} \quad p \leqslant 0. \tag{2}$$

A state, where in addition to the filled negative modes, $m > 0$ ($-m > 0$) consecutive positive (negative) modes are also filled (empty) is denoted by $|m\rangle$ ($|-m\rangle$).

Excited states are obtained by a repeated action of fermionic bilinear operators $\sum A_{pq}\psi_p^\dagger\psi_q$ creating particle-hole excitations on the vacuum. These operators form an infinite-dimensional algebra $gl(\infty)$. Coherent states of this algebra are the states obtained by the action of a group element on the vacuum

$$\langle m|g = \langle m| \exp\left(\sum_{p,q} A_{pq} : \psi_p^\dagger \psi_q :\right). \tag{3}$$

A coherent state represents *particle-hole wave packets*. It is characterized by a infinite matrix A_{pq}. The colon in (3) and throughout the paper denotes normal ordering with respect to the vacuum (2), $: \psi_p^\dagger\psi_q := \psi_p^\dagger\psi_q - \langle 0|\psi_p^\dagger\psi_q|0\rangle$.

Within the algebra of fermionic bilinears one distinguishes two commutative subalgebras. One is generated by positive $k > 0$ (or by negative $k < 0$) modes of the current

$$J_k = \sum_p : \psi_p^\dagger \psi_{p+k} :, \qquad [J_k, J_{-k'}] = k\delta_{kk'}. \tag{4}$$

This subalgebra is central in the construction of [1, 3]. The other commutative subalgebra will be central in the construction of the current paper. It is defined in (13).

An important result [1, 3] is that the matrix element

$$\tau_m(\mathbf{t}) = \langle m|g \exp\left(-\sum_{k>0} t_k J_{-k}\right)|m\rangle \tag{5}$$

considered as a function of an infinite number of continuous parameters $\mathbf{t} = \{t_k\}$ and of an integer parameter m is the τ-function of an integrable hierarchy. τ_m depends also on the state $\langle g|$ (or on the matrix A_{pq}), but this dependence will be omitted in notations throughout the paper. This means that the family of matrix elements $\tau_m(\mathbf{t})$ obeys a bilinear identity:

$$\oint_0 \exp\left(\sum_{k>0}(t_k - t'_k)z^k\right)\tau_m(\mathbf{t} - [z])\tau_{m+n}(\mathbf{t}' + [z])z^n\,\mathrm{d}z = 0, \tag{6}$$

where $\mathbf{t} \pm [z] \equiv \{t_k \pm z^{-k}/k\}\, k = 1, 2, \ldots$ and $\mathbf{t}, \mathbf{t}'$ are two independent sets of parameters. The integral goes along a small contour around the origin.

The bilinear identity (6) encodes an infinite set of hierarchically structured nonlinear differential equations in the variables (flows) t_k. For a given integer n, this set is called the n-MKP (modified KP) hierarchy. The differential equations with respect to the flows t_k in a bilinear (Hirota) form are obtained in each order of the double series expansion of the integrand of (6) in powers of z^{-1} and $t'_k = t_k$.

The KP-hierarchy ($n = 0$) consists of a set of differential equations on τ_m. The first equation of the KP-hierarchy reads

$$\left(D_1^4 + 3D_2^2 - 4D_1D_3\right)\tau_m \cdot \tau_m = 0, \tag{7}$$

where D_k is the Hirota derivative in the variable t_k defined as $D_k^n a \cdot b = \left(\partial_{t_k} - \partial_{t'_k}\right)^n a(\mathbf{t})b(\mathbf{t}')|_{t'_k = t_k}$. This equation is a bilinear form of the Kadomtsev–Petviashvili equation of plasma physics [1]. It gave the name (KP) to the hierarchy.

The MKP-hierarchy involves a discrete evolution in m. The first equation of the 1-MKP-hierarchy reads

$$\left(D_2 + D_1^2\right)\tau_{m+1} \cdot \tau_m = 0. \tag{8}$$

1.2. Evolution in the space of parameters

Some physical applications arise if $t_k = -t^*_{-k}$. We write $t_k = \mathrm{i}\delta t A_{-k}$ and note that the operator $\sum_k t_k J_k = \mathrm{i}\delta t \int A(x)\rho(x)(\mathrm{d}x/2\pi)$ represents the action of the scalar electromagnetic potential $A(x) = \mathrm{i}\sum_k t_{-k}\,\mathrm{e}^{\mathrm{i}kx}$

$$\exp\left(-\sum_k t_k J_k\right) = \exp\left(-\mathrm{i}\delta t \int A(x)\rho(x)\,\mathrm{d}x\right), \qquad \rho(x) =: \psi^\dagger(x)\psi(x) :, \tag{9}$$

which was turned on instantaneously for a short period δt. This matrix element is relevant, e.g., for the process when an electronic system being initially in an excited state $\langle g| = \langle 0|g$ is instantaneously hit by radiation. The τ-function then is the amplitude of the probability of finding a system in the vacuum (ground) state.

An alternative interpretation of the formal expression (5), common in the literature, emphasizes the commutativity of flows with respect to parameters t_k. In this interpretation the operator $\exp\left(\sum_k t_k J_{-k}\right)$ is viewed as an evolution operator with a set of 'times' t_k corresponding to a family of mutually commuting 'Hamiltonians' J_k. The excited state

$\langle g| = \langle m|g$ evolves in 'times'. The τ-function is an overlap between the 'evolved' excited state $\langle g(\mathbf{t})|$ and the vacuum

$$\tau_m = \langle g(\mathbf{t})|m\rangle, \qquad \langle g(\mathbf{t})| = \langle m|g \exp\left(-\sum_k t_k J_{-k}\right). \tag{10}$$

This terminology, although common, is somewhat confusing. Of course, the 'Hamiltonians', J_k are not Hermitian and there is no energy associated with this evolution. The 'evolution' occurs in the space of parameters. This may be the reason why application of the powerful mathematical apparatus of soliton theory to electronic physics has been limited.

In the next subsection we introduce the τ-function as a matrix element of the real-time evolution generated by the Hermitian Hamiltonian (1).

1.3. Evolution in real time

In physics one is interested in the real-time evolution driven by a Hermitian positive Hamiltonian as (1) rather than currents (4).

Can the theory of solitons be used to study the dynamics of fermions in real spacetime, when the evolution is described by a unitary operator

$$\mathrm{e}^{-\mathrm{i}Ht+\mathrm{i}Px} \tag{11}$$

corresponding to a positive energy (1) and momentum

$$P = \sum_{p\in I} p : \psi_p^\dagger \psi_p : . \tag{12}$$

In this case the vacuum is selected as a ground state of the Hamiltonian

$$H|0\rangle = 0.$$

Indeed, the set of Hermitian Hamiltonians

$$H_k = \sum_{p\in I} p^k : \psi_p^\dagger \psi_p :, \qquad k = 1, 2, \ldots \tag{13}$$

generates another commutative subalgebra of fermionic bilinears. Here we denote the set of quantized momenta for the system by I.

Surprisingly, this question has never been addressed systematically. The most relevant papers we know are [4, 5] where the relation of time-dependent Green functions of impenetrable bosons and Green functions of the Ising model to the Painlevé equations has been established.

In this paper we will show that, indeed, an action of the commutative Hamiltonians (13) generates an integrable hierarchy. Specifically, we show that the matrix element

$$\tau_m(\mathbf{t}) = \langle 0|g \exp\left(-\sum_{k>0} t_k H_k\right) : \mathrm{e}^{-(a+m)\varphi(x)} : |0\rangle, \qquad 0 < |a| < 1/2, \qquad m \text{ is integer} \tag{14}$$

obeys the MKP-hierarchy[3].

Namely, the τ-functions $\tau_m(\mathbf{t})$, similar to (5), satisfy a set of equations analogous to (7), (8). These equations are generated by the identity:

[3] Here we take $a \neq 0$. Although the τ-function obeys the same a-independent equations, it degenerates at $a = 0$ and can be computed by elementary means. The matrix element $\langle 0|g \exp\left(-\sum_{k>0} t_k H_k\right) : \mathrm{e}^{-(a+m)\varphi(x)} : \exp\left(\sum_{k>0} t_k H_k\right) h|0\rangle$, where h is an exponent in fermionic bilinears, can be treated by similar means [14], and in fact obeys the same integrable hierarchy. We choose to discuss only (14) in this paper for the sake of simplicity.

$$\oint_0 \exp\left(\sum_{k>0}(t_k - t'_k)z^k\right)\frac{\Gamma(z+m+n)}{\Gamma(z+m)}\tau_m(\mathbf{t}-[z])\tau_{m+n}(\mathbf{t}'+[z])\,\mathrm{d}z = 0, \quad (15)$$

which is a modified form of (6). The factor $\frac{\Gamma(z+m+n)}{\Gamma(z+m)} = \prod_{j=m}^{m+n-1}(z+j)$ replaces z^n in (6)[4].

In equation (14) the operator φ is defined as a Bose field

$$\varphi(x) = \sum_{k\neq 0}\frac{1}{k}\,\mathrm{e}^{\mathrm{i}kx}J_k + \mathrm{i}J_0 x \quad (16)$$

and

$$:\mathrm{e}^{a\varphi(x)} := \mathrm{e}^{a\varphi_-(x)}\,\mathrm{e}^{\mathrm{i}axJ_0}\,\mathrm{e}^{a\varphi_+(x)}, \quad (17)$$

where $\varphi_\pm(x) = \sum_{\pm k>0}(\mathrm{e}^{\mathrm{i}kx}/k)J_k$ contain only positive or negative modes respectively. Note that the flow t_1 just shifts the coordinate x by $\mathrm{i}t_1$ as $\mathrm{e}^{t_1H_1}\varphi(x)\,\mathrm{e}^{-t_1H_1} = \varphi(x+\mathrm{i}t_1)$, so we may set $x = 0$ in some formulae below. In physical applications we identify

$$\text{time}: t = \mathrm{i}t_2, \qquad \text{space coordinate in the Galilean frame}: x - v_F t = -\mathrm{i}t_1, \quad (18)$$

where $v_F = p_F/M$ is a Fermi velocity.

The operator $:\mathrm{e}^{a\varphi(x)}:$ in (14) is called *a boundary condition changing operator*. This operator acting on a vacuum creates *a boundary state*

$$|B_m(x)\rangle =: \mathrm{e}^{-(a+m)\varphi(x)}:|0\rangle. \quad (19)$$

The meaning of the boundary condition changing operator is clarified by the action of the counting operator $n(y) = \int_0^y(:\psi^\dagger(y')\psi(y'): -(a+m))\,\mathrm{d}y'$ on the boundary state

$$\langle B_m(x)|n(y)|B_m(x)\rangle = (a+m) \quad \text{if} \quad 0 < x < y, \quad \text{or} \quad 0 \quad \text{otherwise}. \quad (20)$$

In other words, the boundary condition changing operator moves particles towards the point x creating an increase of the density $\rho(y) = \partial_y n(y)$ at that point.

Boundary states play an important role in 2D-critical phenomena in systems with boundaries. The connection of the latter to 1D fermions is following. The fields (e.g., $\psi(x)$ and $\varphi(x)$) are originally defined on the unit circle. One views the circle as a contour embedded in the complex plane z. The fields and their differentials can be analytically extended to the exterior of the circle and to its interior. Instead of considering fields defined at the point z' in the interior of the circle one views them as defined at the point $z = 1/\bar{z}'$ of the circle's exterior (Schwarz reflection). From this point of view the circle $z = \mathrm{e}^{-\mathrm{i}x}$, $x \in \mathbb{R}$, is seen as a boundary of the holomorphic 2D-field theory defined in its exterior.

In electronic physics boundary states appear in numerous problems related to the Fermi-edge singularities, quantum impurity problems and other phenomena related to orthogonality catastrophe ([10], see, e.g. [8]). We will discuss some applications to electronic physics elsewhere [14].

In terms of the boundary state, our τ-function (14) reads

$$\tau_m(\mathbf{t}) = \langle G(\mathbf{t})|B_m\rangle, \qquad \langle G(\mathbf{t})| = \langle 0|g\exp\left(-\sum_{k>0}t_kH_k\right). \quad (21)$$

The τ-function depends on two parameters: a fraction a, and an integer m. Below we fix a and consider m as a running parameter.

One can think of this matrix element as follows. An excited coherent state $\langle g|$ evolves up to a time t and then is measured at a point x by a projection onto a boundary state $|B_a(x)\rangle$.

[4] The difference between (15) and (6) is inessential insofar as the former may be transformed into the latter by the transformation $\tau_m \to \tau_m\exp(\sum_{k\geqslant 0}V(m+k))$, where $V(z) = \sum_k t_kz^k$, however, this transformation changes analytical properties of the τ-function as a function of the 'times'.

A comment is in order. The presence of the boundary operator : $\mathrm{e}^{a\varphi(x)}$: changes the momentum quantization. The eigenvalues p of the momentum operator (12) appearing in (12), (13) are no longer integer, but are integers shifted by a:

$$p \in \mathbb{Z} + a, \tag{22}$$

that is I in (12) and (13) is given by $I = \mathbb{Z} + a$.

An obvious generalization of this result is a passage from the KP to the 2D-Toda hierarchy [9]. The matrix element

$$\langle B_m | \exp\left(-\sum_k \bar{t}_k H_k\right) g \exp\left(-\sum_k t_k H_k\right) |B_m\rangle$$

obeys the 2D-Toda hierarchy with respect to two sets of parameters t_k and $\bar{t}_k$ and a discrete parameter m. Other obvious generalization occurs when fermions are placed on a spatial lattice so that the energy spectrum is a periodic function of momentum. We do not discuss these generalizations here.

The modified bilinear identity (15) generates an infinite set of Hirota bilinear equations. It, therefore, obeys the KP-hierarchy with respect to infinitely many flow parameters t_k. It also obeys the MKP-hierarchy with respect to a discrete parameter m characterizing the boundary state. In a special case when only two physical flow parameters (18) t_2 and t_1 are present we obtain the 1-MKP equation. In the Galilean frame it reads

$$\left(\mathrm{i}D_t + D_x^2 - 2\mathrm{i}m D_x\right)\tau_{m+1} \cdot \tau_m = 0. \tag{23}$$

The independent variables x and t in this equation are physical space and time coordinates of a 1D fermionic system. Among the hierarchy, this equation seems to be the most important for the study of the dynamics of electronic systems. The two functions τ_m and τ_{m+1} are not independent but are connected by certain analytical properties, thus making equation (23) closed. In the most interesting situations (corresponding to particular choices of coherent state g) the analytical conditions are explicit and correspond to a certain reduction of the MKP-hierarchy.

We note here that the matrix element (14) give solutions of the MKP-hierarchy, which are essentially different from those given by (5). For example, we show below that (14) has only positive Fourier modes with respect to all odd flows $\mathrm{i}t_1, \mathrm{i}t_3, \ldots$. This analytical condition is restrictive. In particular, it does not admit soliton solutions. In contrast, it is well known that (5) can generate soliton solutions (see section 3).

Some mathematical constructions related of the matrix element (14), although devoted to different problems, have appeared recently in papers by A Orlov [6] and by A Okounkov, R Pandharipande and N A Nekrasov [7].

2. Hirota equation and bosonization

Before we proceed, we present another form of the bilinear identity (15) for a particular case $n = 1$. Setting $\mathbf{t} = \mathbf{t}^{(0)} + [p], \mathbf{t}' = \mathbf{t}^{(0)} - [q]$ the factor $(z+m)\exp\left(\sum_k (t_k - t'_k)z^k\right)$ becomes $\frac{(z+m)pq}{(q-z)(p-z)}$. The integrand has three poles at $z = 0, p, q, \infty$ and can be computed. It gives the discrete Hirota equation

$$(p+m)\tau_m \cdot \tau_{m+1}([p]-[q]) - (q+m)\tau_m([p]-[q]) \cdot \tau_{m+1} \\ - (p-q)\tau_m([p]) \cdot \tau_{m+1}(-[q]) = 0, \tag{24}$$

where we have dropped the term $\mathbf{t}^{(0)}$ in each of the arguments of the τ functions. We stress that p and q are shifted integers $p, q \in \mathbb{Z} + a$.

Hirota's equation, written in the discrete form (24) appears as a particular specification of the integral form of the bilinear identity (6). In fact, they carry the same information as an integral form of the bilinear identity written for an arbitrary n. The latter can be obtained from the former [2]. In the rest of this section we will derive the discrete form of Hirota's equation (24) for τ-functions defined in (14). It will be equivalent to proving (15).

Our main equation (23) directly follows from Hirota equation written in the form (24). It appears in the leading order expansion at large q and p, and setting $q = -p$.

2.1. Bosonization in coordinate space and the τ-function $\tau_m = \langle g(\mathbf{t})|m\rangle$

First, we show how to derive the discrete Hirota equation (25) for the conventional matrix element matrix (5). Then we derive the discrete Hirota equation (24) for our τ-function (14). The first derivation is well known [1–3, 12]. We, nevertheless, present it here in order to use it as a guide for the following derivation of Hirota's equation for the real-time evolution. The comparison of these two derivations is instructive.

Applying the same trick as is the previous section to (6) we obtain a discrete Hirota equation for the τ-function (5)

$$p\tau_m \cdot \tau_{m+1}([p]-[q]) - q\tau_m([p]-[q]) \cdot \tau_{m+1} - (p-q)\tau_m([p]) \cdot \tau_{m+1}(-[q]) = 0. \tag{25}$$

To derive it one needs two technical ingredients: *bosonization* and the *Wick's theorem*.

Bosonization formulae. The Bosonization formulae express the action of fermion operators in terms of bosons. We will need the following formulae (briefly derived below)

$$\psi^\dagger(x)\psi(y)|m\rangle = \frac{1}{1-\mathrm{e}^{\mathrm{i}(x-y)}} : \mathrm{e}^{\varphi(y)-\varphi(x)} : |m\rangle, \tag{26}$$

$$\psi(x)|m+1\rangle = \mathrm{e}^{\mathrm{i}x} : \mathrm{e}^{\varphi(x)} : |m\rangle, \qquad \psi^\dagger(x)|m\rangle =: \mathrm{e}^{-\varphi(x)} : |m+1\rangle. \tag{27}$$

Let us now use these bosonization formulae to 'fermionize' the τ-function $\tau_{m+1}([p]-[q])$. We set $q = \mathrm{e}^{\mathrm{i}x}$ and $p = \mathrm{e}^{\mathrm{i}y}$, and write the τ-function $\tau_{m+1}([p]-[q]) = \langle m+1|g\,\mathrm{e}^{-\sum_k t_k^{(0)} J_{-k}} : \mathrm{e}^{\varphi(y)-\varphi(x)} : |m+1\rangle$, appearing in (25), in terms of fermion operators:

$$\begin{aligned} &\tau_{m+1}([p]-[q]) = \mathrm{e}^{\mathrm{i}(m+1)(x-y)}(1-\mathrm{e}^{\mathrm{i}(x-y)})\langle m+1|g(\mathbf{t})\psi^\dagger(x)\psi(y)|m+1\rangle, \\ &g(\mathbf{t}) = g\exp\left(-\sum_k t_k^{(0)} J_{-k}\right). \end{aligned} \tag{28}$$

Wick's theorem. We apply Wick's theorem to 4-fermion matrix elements. Let χ_i be a linear combination of fermionic modes $\chi_i = \sum_q v_q^{(i)}\psi_q$, and $|G_2\rangle$ and $\langle G_1|$ are coherent states (i.e., states of the form (3)). The general form of Wick's theorem for 4-fermion matrix elements reads

$$\begin{aligned} \langle G_1|\chi_1^\dagger\chi_2\chi_3^\dagger\chi_4|G_2\rangle\langle G_1|G_2\rangle &= \langle G_1|\chi_1^\dagger\chi_2|G_2\rangle\langle G_1|\chi_3^\dagger\chi_4|G_2\rangle \\ &\quad + \langle G_1|\chi_1^\dagger\chi_4|G_2\rangle\langle G_1|\chi_2\chi_3^\dagger|G_2\rangle. \end{aligned} \tag{29}$$

A proof of the Hirota bilinear identity (25). We apply Wick's theorem to the matrix element

$$\langle m|\psi_{m+1}g(\mathbf{t})\psi^\dagger(x)\psi(y)\psi^\dagger_{m+1}|m\rangle\langle m|g(\mathbf{t})|m\rangle = \langle m+1|g(\mathbf{t})|m+1\rangle\langle m|g(\mathbf{t})\psi^\dagger(x)\psi(y)|m\rangle + \langle m+1|g(\mathbf{t})\psi^\dagger(x)|m\rangle\langle m|g(\mathbf{t})\psi(y)|m+1\rangle. \quad (30)$$

Equation (25) emerges if one writes each of the matrix elements appearing in this identity as τ-functions with shifted arguments, by further use of the bosonization formulae (26), (26).[5]

Derivation of bosonization formulae (26), (27). It will be instructive to have a short proof of bosonization formulae (26), (27) as we are going to use a similar approach in the next section. In the following we set for simplicity $m = 0$.

Let us project the rhs and the lhs of (26) onto a general state with a given number of particles $\langle\{x_i, y_i\}| = \langle 0|\big(\prod_{i=1}^N \psi^\dagger(x_i)\psi(y_i)\big)$, consisting of N particle and N hole excitations with coordinates x_i, y_i, and compare the results.

First of all, let us use Wick's theorem to calculate the overlap between the state with N particle-hole pairs and the vacuum. This overlap is a determinant of a single-particle Green's function $\langle 0|\psi^\dagger(x)\psi(y)|0\rangle = (1-\mathrm{e}^{\mathrm{i}(x-y)})^{-1}$

$$\langle\{x_i, y_i\}|0\rangle = \exp\left(\mathrm{i}\sum_{i=1}^N y_i\right)\det_{i,j\leqslant N}\left(\frac{1}{\mathrm{e}^{\mathrm{i}y_i}-\mathrm{e}^{\mathrm{i}x_j}}\right) = \exp\left(\mathrm{i}\sum_{i=1}^N y_i\right)\frac{\prod_{i<j\leqslant N}(\mathrm{e}^{\mathrm{i}y_i}-\mathrm{e}^{\mathrm{i}y_j})(\mathrm{e}^{\mathrm{i}x_j}-\mathrm{e}^{\mathrm{i}x_i})}{\prod_{i,j\leqslant N}(\mathrm{e}^{\mathrm{i}y_i}-\mathrm{e}^{\mathrm{i}x_j})}, \quad (31)$$

where we used the multiplicative form of the Cauchy determinant

$$\det_{i,j\leqslant N}\left(\frac{1}{q_i-p_j}\right) = \frac{\prod_{i<j\leqslant N}(q_i-q_j)(p_j-p_i)}{\prod_{i,j\leqslant N}(q_i-p_j)}. \quad (32)$$

The lhs of (26) is computed by extending the previous formula to $N+1$ particle-hole pairs and setting $x_{N+1} = x$ and $y_{N+1} = y$

$$\langle\{x_i, y_i\}|\psi^\dagger(x)\psi(y)|0\rangle = \frac{\mathrm{e}^{\mathrm{i}y}}{\mathrm{e}^{\mathrm{i}y}-\mathrm{e}^{\mathrm{i}x}}\prod_{i\leqslant N}\frac{(\mathrm{e}^{\mathrm{i}y}-\mathrm{e}^{\mathrm{i}y_i})(\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{\mathrm{i}x_i})}{(\mathrm{e}^{\mathrm{i}y}-\mathrm{e}^{\mathrm{i}x_i})(\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{\mathrm{i}y_i})}\langle\{x_i, y_i\}|0\rangle. \quad (33)$$

The rhs of (26) can be computed by (i) noticing that $\mathrm{e}^{b\varphi}|0\rangle = \mathrm{e}^{b\varphi_-}|0\rangle$, (ii) pulling the vertex operator $\mathrm{e}^{b\varphi_-}$ to the left using the identity

$$\mathrm{e}^{-b\varphi_-(x)}\psi^\dagger(x_i)\psi(y_i)\,\mathrm{e}^{b\varphi_-(x)} = \left(\frac{\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{\mathrm{i}y_i}}{\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{\mathrm{i}x_i}}\right)^b\psi^\dagger(x_i)\psi(y_i) \quad (34)$$

and then (iii) using $\langle 0|\,\mathrm{e}^{b\varphi_-(x)} = 0$. As a result we get

$$\langle\{x_i, y_i\}|:\mathrm{e}^{\varphi(y)-\varphi(x)}:|0\rangle = \prod_{i\leqslant N}\frac{(\mathrm{e}^{\mathrm{i}y}-\mathrm{e}^{\mathrm{i}y_i})(\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{\mathrm{i}x_i})}{(\mathrm{e}^{\mathrm{i}y}-\mathrm{e}^{\mathrm{i}x_i})(\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{\mathrm{i}y_i})}\langle\{x_i, y_i\}|0\rangle. \quad (35)$$

Comparing equations (35) and (33) gives the bosonization formula (26). Equation (27) can be derived in a similar way, or by setting $x = -\mathrm{i}\infty$ in (26) and using the fact that $\psi^\dagger(-\mathrm{i}\infty)|m\rangle = |m+1\rangle$. In all formulae x and y carry small positive imaginary parts.

[5] This proof appeals to a general view on τ-functions as Plücker coordinates of the Grassmann manifold [2].

2.2. Bosonization in momentum space and the τ-function $\tau_m = \langle G(\mathbf{t})|B_m\rangle$

Now we show that the matrix element (14) obeys the Hirota equation (24). The steps of the derivation in this section follow the steps of the previous section, but technically different. We repeat that in this section all momenta are shifted integers $p, q \in \mathbb{Z} + a$.

The important property of the boundary state is displayed by the form of its wave-function in momentum space. We have $\langle 0|B_m\rangle = 1$ and two important formula

$$\langle 0|\psi_q^\dagger \psi_p|B_m\rangle = f_m^+(p)\frac{1}{p-q} f_m^-(q), \qquad q \leqslant a < p, \tag{36}$$

$$\frac{p+m}{q+m}\langle 0|\psi_q^\dagger \psi_p|B_m\rangle = \langle 0|\psi_q^\dagger \psi_p|B_{m+1}\rangle, \tag{37}$$

where

$$\begin{aligned} f_m^+(p) &= \frac{\Gamma(p+m)}{\Gamma(m+a)\Gamma(p-a)}, \\ f_m^-(q) &= \frac{\Gamma(1-q-m)}{\Gamma(1-m-a)\Gamma(1-q+a)}. \end{aligned} \tag{38}$$

The matrix element (36) is also non-zero at $p = q \leqslant a$, where it is equal to 1.[6] Equation (37) follows from (36), (38) with the help of the formula

$$f_m^+(p) f_m^-(q)\frac{p+m}{q+m} = f_{m+1}^+(p) f_{m+1}^-(q). \tag{39}$$

We prove these formulae at the end of this section.

Formula (36) is related to a phenomenon referred to as 'orthogonality catastrophe' [10]. Namely, if the length of the ring is large, i.e, in the limit $p, -q \gg |m|$ formula (36) reads

$$\langle 0|\psi_q^\dagger \psi_p|B_m\rangle = (-1)^m \frac{\sin(\pi a)}{\pi}\frac{1}{p-q}\left(\frac{p}{-q}\right)^{a+m}. \tag{40}$$

The overlap between the excited state and the boundary state vanishes as the momentum of the excited particle approaches the Fermi-point $\langle 0|\psi_q^\dagger \psi_p|B_m\rangle \sim p^{a+m}$ as $p \to 0$.

Consider now a generic excited state $\langle \{p_i, q_i\}| = \langle 0| \prod_{i=1}^N \psi_{q_i}^\dagger \psi_{p_i}$ characterized by a set of N-particles with momenta $p_i > a$ and N holes with momenta $q_i \leqslant a$. The overlap of this excited state with the boundary state is obtained from Wick's theorem and equation (36)

$$\begin{aligned} \langle \{p_i, q_i\}|B_m\rangle &= \det_{i,j\leqslant N} \langle 0|\psi_{q_j}^\dagger \psi_{p_i}|B_m\rangle = \det_{i,j\leqslant N}\left(f_m^+(p_i)\frac{1}{p_i-q_j} f_m^-(q_j)\right) \\ &= \prod_{i\leqslant N} f_m^+(p_i) f_m^-(q_i)\frac{\prod_{i<j\leqslant N}(p_i-p_j)(q_j-q_i)}{\prod_{i,j\leqslant N}(p_i-q_j)}, \end{aligned} \tag{41}$$

where in the last line we again used the Cauchy determinant formula (32).

Let us note that up to the multiplicative factor $\prod_i f_m^+(p_i) f_m^-(q_i)$, the overlap (41) between the excited state and the boundary state written in momentum space is given by the Cauchy determinant, i.e., has the same form as the similar overlap written in real space (31). The determinants become identical under a substitution $p_i = \mathrm{e}^{\mathrm{i}x_i}$ and $q_i = \mathrm{e}^{\mathrm{i}y_i}$.

[6] Formula (36) is understood as being analytically continued from the real axis to the complex plane of p and q. In particular, at $p = q$ it should be understood as a limit $p \to q$ at fixed q.

We observe an interesting phenomenon: the boundary state $|B_m\rangle$ interchanges momentum and coordinate spaces. The transmutation

$$\text{coordinate space} \leftrightarrow \text{momentum space}, \tag{42}$$

$$\text{Fermi sea} \leftrightarrow \text{boundary state} \tag{43}$$

is the reason why different matrix elements (5), (14) obey similar nonlinear equations. There is no one-to-one correspondence between coordinate and momentum space—the space is continuous and finite while the momentum space is discrete and infinite.

Following the logic of the previous section we proceed in two steps. First, we establish the bosonization formula in momentum space. They will allow us to rewrite τ-functions in equation (24) as matrix elements of fermionic operators. Then equation (24) is equivalent to a particular form of Wick's theorem.

Bosonization formulae. Let us introduce a bosonic field associated with the set of commuting Hamiltonians (13)

$$\Phi(s) = \sum_{k>0} \frac{s^{-k}}{k} H_k = -\sum_p \ln\left(1 - \frac{p}{s}\right) : \psi_p^\dagger \psi_p : . \tag{44}$$

Bosonization in the momentum space is summarized by the formulae

$$\psi_q^\dagger \psi_p |B_m\rangle = \frac{f_m^+(p) f_m^-(q)}{p-q}\, \mathrm{e}^{\Phi(q)-\Phi(p)} |B_m\rangle, \tag{45}$$

and

$$\lim_{p\to -m} (p+m) \psi_q^\dagger \psi_p |B_m\rangle = (-1)^m \frac{\sin \pi a}{\pi} f_{m+1}^-(q)\, \mathrm{e}^{\Phi(q)} |B_{m+1}\rangle. \tag{46}$$

These formula will be derived below. We remark here that (45), (46) should be understood as analytically continued formula for matrix elements. For example, one should first calculate the overlap of the lhs of (46) with some bra-state, then analytically continue it from $p \in \mathbb{Z}+a$ to the complex p-plane and only then take a limit $p \to -m$.

Once bosonization formulae in momentum space (45), (46) are established we are ready to prove the Hirota equation (24).

Wick's theorem. First, we apply the Wick theorem to the 4-fermion matrix element

$$\begin{aligned}\langle G(\mathbf{t})|\psi_q^\dagger \psi_p \psi_Q^\dagger \psi_P |B_m\rangle \langle G(\mathbf{t})|B_m\rangle &= \langle G(\mathbf{t})|\psi_q^\dagger \psi_p |B_m\rangle \langle G(\mathbf{t})|\psi_Q^\dagger \psi_P |B_m\rangle \\ &\quad - \langle G(\mathbf{t})|\psi_q^\dagger \psi_P |B_m\rangle \langle G(\mathbf{t})|\psi_Q^\dagger \psi_p |B_m\rangle.\end{aligned} \tag{47}$$

Next we multiply both sides by $\frac{P+m}{f_{m+1}^-(Q)}$, send $P \to -m$, and use equation (46). We get

$$\begin{aligned}&\frac{q-Q}{p-Q} \langle G(\mathbf{t})|\, \mathrm{e}^{\Phi(Q)} \psi_q^\dagger \psi_p |B_{m+1}\rangle \langle G(\mathbf{t})|B_m\rangle \\ &\quad = \langle G(\mathbf{t})|\psi_q^\dagger \psi_p |B_m\rangle \langle G(\mathbf{t})|\, \mathrm{e}^{\Phi(Q)} |B_{m+1}\rangle \\ &\qquad - \frac{f_{m+1}^-(q)}{f_{m+1}^-(Q)} \langle G(\mathbf{t})|\, \mathrm{e}^{\Phi(q)} |B_{m+1}\rangle \langle G(\mathbf{t})|\psi_Q^\dagger \psi_p |B_m\rangle.\end{aligned} \tag{48}$$

The next step is to use the bosonization formula (45)

$$\begin{aligned}&\frac{q-Q}{p-Q} \frac{p+m}{q+m} \langle G(\mathbf{t})|\, \mathrm{e}^{\Phi(Q)}\, \mathrm{e}^{\Phi(q)-\Phi(p)} |B_{m+1}\rangle \langle G(\mathbf{t})|B_m\rangle \\ &\quad = \langle G(\mathbf{t})|\, \mathrm{e}^{\Phi(q)-\Phi(p)} |B_m\rangle \langle G(\mathbf{t})|\, \mathrm{e}^{\Phi(Q)} |B_{m+1}\rangle \\ &\qquad - \frac{Q+m}{q+m} \frac{p-q}{p-Q} \langle G(\mathbf{t})|\, \mathrm{e}^{\Phi(q)} |B_{m+1}\rangle \langle G(\mathbf{t})|\, \mathrm{e}^{\Phi(Q)-\Phi(p)} |B_m\rangle.\end{aligned} \tag{49}$$

We recognize the matrix elements in (49) as τ-functions (14) with shifted arguments, e.g., $\langle G(\mathbf{t})|\,\mathrm{e}^{\Phi(q)}|B_{m+1}\rangle = \tau_{m+1}(-[q])$. The last step is to take a limit $Q\to\infty$. In this limit $\Phi(Q)\to 0$ and (49) brings us to (24), which completes the proof.

Derivation of formula for matrix elements (36), (38). Now we prove formulae (36), (38). First, we find using (34) that in coordinate representation

$$\langle 0|\psi^\dagger(x)\psi(y)|B_m\rangle = \langle 0|\psi^\dagger(x)\psi(y)|0\rangle\left(\frac{1-\mathrm{e}^{\mathrm{i}x}}{1-\mathrm{e}^{\mathrm{i}y}}\right)^{m+a} = \frac{\mathrm{e}^{\mathrm{i}y}}{\mathrm{e}^{\mathrm{i}y}-\mathrm{e}^{\mathrm{i}x}}\left(\frac{1-\mathrm{e}^{\mathrm{i}x}}{1-\mathrm{e}^{\mathrm{i}y}}\right)^{m+a}. \tag{50}$$

We consider $(p-q)\langle 0|\psi_q^\dagger\psi_p|B_m\rangle$ and pass to the coordinate representation $(p-q)\langle 0|\psi_q^\dagger\psi_p|B_m\rangle\to(\mathrm{i}\partial_x+\mathrm{i}\partial_y)\langle 0|\psi^\dagger(x)\psi(y)|B_m\rangle$. Then we calculate

$$(\mathrm{i}\partial_x+\mathrm{i}\partial_y)\left[\frac{\mathrm{e}^{\mathrm{i}y}}{\mathrm{e}^{\mathrm{i}y}-\mathrm{e}^{\mathrm{i}x}}\left(\frac{1-\mathrm{e}^{\mathrm{i}x}}{1-\mathrm{e}^{\mathrm{i}y}}\right)^{m+a}\right] = (m+a)(1-\mathrm{e}^{\mathrm{i}x})^{m+a-1}\,\mathrm{e}^{\mathrm{i}y}(1-\mathrm{e}^{\mathrm{i}y})^{-(a+m+1)}.$$

We observe that the right-hand side factorizes. Taking an inverse Fourier transform we obtain (36) with

$$f_m^-(q) = \int_0^{2\pi}\mathrm{e}^{\mathrm{i}qx}(1-\mathrm{e}^{\mathrm{i}x})^{m+a-1}\frac{\mathrm{d}x}{2\pi},$$
$$f_m^+(p) = (m+a)\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}(p-1)y}(1-\mathrm{e}^{\mathrm{i}y})^{-(m+a+1)}\frac{\mathrm{d}y}{2\pi}.$$

The calculation of these integrals gives (38) for $f_m^\pm$. These formula can also be found in [11].

Derivation of bosonization formula (45), (46). To prove formula (45) we evaluate the projection of its rhs on the state $\langle\{p_i,q_i\}|$. With the help of the formula

$$\mathrm{e}^{b\Phi(s)}\psi_q^\dagger\psi_p\,\mathrm{e}^{-b\Phi(s)} = \left(\frac{s-p}{s-q}\right)^b\psi_q^\dagger\psi_p \tag{51}$$

analogous to (34) and using $\langle 0|\Phi(s)=0$ we obtain

$$\langle\{p_i,q_i\}|\,\mathrm{e}^{\Phi(q)-\Phi(p)}|B_m\rangle = \langle\{p_i,q_i\}|B_m\rangle\prod_{i\leqslant N}\frac{(p-p_i)(q-q_i)}{(p-q_i)(q-p_i)}. \tag{52}$$

On the other hand, using the multiplicative representation for the lhs of (45) similar to (41) we obtain

$$\begin{aligned}\langle\{p_i,q_i\}|\psi_q^\dagger\psi_p|B_m\rangle &= \det_{i,j\leqslant N+1}\left(f_m^+(p_i)\frac{1}{p_i-q_j}f_m^-(q_j)\right)\\ &= \langle\{p_i,q_i\}|B_m\rangle\frac{f_m^+(p)f_m^-(q)}{p-q}\prod_{i\leqslant N}\frac{(p-p_i)(q-q_i)}{(p-q_i)(q-p_i)},\end{aligned} \tag{53}$$

where $p_{N+1}=p$ and $q_{N+1}=q$. Comparing (53) and (52) we prove (45).

To obtain (46), we first use (39) to obtain

$$\langle\{p_i,q_i\}|B_{m+1}\rangle = \langle\{p_i,q_i\}|B_m\rangle\prod_{i\leqslant N}\frac{p_i+m}{q_i+m}. \tag{54}$$

Then we use (54) and (53) to proceed as follows:

$$\begin{aligned}\langle\{p_i,q_i\}|\,\mathrm{e}^{\Phi(q)}|B_{m+1}\rangle &= \prod_{i\leqslant N}\frac{q-q_i}{q-p_i}\langle\{p_i,q_i\}|B_{m+1}\rangle = \prod_{i\leqslant N}\frac{q-q_i}{q-p_i}\frac{p_i+m}{q_i+m}\langle\{p_i,q_i\}|B_{m+1}\rangle\\ &= \langle\{p_i,q_i\}|\psi_q^\dagger\psi_p|B_m\rangle\frac{p-q}{f_m^+(p)f_m^-(q)}\prod_{i\leqslant N}\frac{p_i+m}{p_i-p}\frac{q_i-p}{q_i+m}.\end{aligned} \tag{55}$$

The rhs of (55) effectively does not depend on p and we take its limit as $p \to -m$. The product in the last line disappears and we conclude that

$$\mathrm{e}^{\Phi(q)}|B_{m+1}\rangle = -\frac{q+m}{f_m^-(q)} \lim_{p\to -m} \frac{1}{f_m^+(p)} \psi_q^\dagger \psi_p |B_m\rangle. \tag{56}$$

Finally, using

$$\lim_{p\to -m} (p+m) f_m^+(p) = (-1)^{m+1}(m+a)\frac{\sin(\pi a)}{\pi}, \tag{57}$$

we arrive at (46).

3. Multi-periodic solutions

In this section we present explicit formulae for multi-phase solutions evolving in real time. These formulae reveal the structure of the τ-function and provide another direct proof of Hirota's equation.

A set of excited states $|\{p_i, q_i\}\rangle$ characterized by momenta of N particles $p_i > a$ and holes $q_i \leqslant a$, $i = 1, \ldots, N$ form an orthonormal basis for the Fock space of fermions. In this basis the Hamiltonians (13) are diagonal: $H_k|\{p_i, q_i\}\rangle = \sum_i \left(p_i^k - q_i^k\right)|\{p_i, q_i\}\rangle$. Using this basis, we decompose

$$\tau_m = \sum_{N\geqslant 0} \sum_{\{p_i, q_i\}} \exp\left(-\sum_{k,i} t_k\left(p_i^k - q_i^k\right)\right) \langle g|\{p_i, q_i\}\rangle\langle\{p_i, q_i\}|B_m\rangle. \tag{58}$$

The matrix elements entering this expression are determinants of two-particle matrix elements. The first one is

$$\langle g|\{p_i, q_i\}\rangle = \det_{i,j} A_{p_i, q_j}, \qquad A_{p,q} = \langle g|p, q\rangle. \tag{59}$$

The second one is the Schur function given by equation (41)

$$\langle\{p_i, q_i\}|B_m\rangle = \det_{i,j\leqslant N}\left(\frac{1}{p_i - q_j}\right) \prod_{i\leqslant N} f_m^+(p_i) f_m^-(q_i). \tag{60}$$

Introducing

$$s_m(\{p_i, q_i\}, g) = \det_{i,j}\left(A^{(m)}_{p_i, q_j}\right),$$

where

$$A^{(m)}_{p,q} = f_m^+(p) A_{p,q} f_m^-(q)$$

we write the τ-function in the form

$$\tau_m = \sum_{N\geqslant 0} \sum_{\{p_i, q_i\}} \exp\left(-\sum_{k,i} t_k\left(p_i^k - q_i^k\right)\right) s_m(\{p_i, q_i\}, g) \det_{ij}\left(\frac{1}{p_i - q_j}\right). \tag{61}$$

It is obvious from this form and inequality $p_i > a \geqslant q_i$ that the τ-function has only positive Fourier components with respect to all odd flows $\mathrm{i}t_1, \mathrm{i}t_2, \ldots$. Indeed, for all odd k we have $p_i^k - q_i^k > 0$. The bilinear identity (15) follows directly from this representation if one uses identities for Schur functions and the property

$$\frac{s_{m+1}}{s_m} = \prod_i \left(\frac{p_i + m}{q_i + m}\right). \tag{62}$$

We do not present this proof of (24) here[7].

[7] An observation that the series (61) for (14) obeys KP-hierarchy, i.e., equation (15) at $n = 0$ has been made in [6].

If the matrix A_{pq} defining the state $\langle g|$ by (59) is a finite matrix, the sum in (61) truncates. It represents a multi-phase solution. In this case the determinant formula holds

$$\tau_m(\mathbf{t}, a) = \det_{i,j\leqslant N}\left(\delta_{ij} + f_m^+(p_i)\,\mathrm{e}^{-\epsilon(p_i)}\frac{A_i}{p_i - q_j}\,\mathrm{e}^{\epsilon(q_j)}\,f_m^-(q_j)\right), \qquad \epsilon(p) = \sum_k t_k p^k. \tag{63}$$

Here $A_i = A_{p_i q_i}$. Note that a similar determinant representation holds for (5). Namely,

$$\tau_m(\mathbf{t}) = \det_{i,j\leqslant N}\left(\delta_{ij} + p_i^{-m}\,\mathrm{e}^{-\epsilon(p_i)}\frac{A_i}{p_i - q_j}\,\mathrm{e}^{\epsilon(q_j)}q_j^{m+1}\right), \qquad \epsilon(p) = \sum_k t_k p^k. \tag{64}$$

Although (63) and (64) are very similar there is an essential difference between them. Namely, in (63) the parameters p_i and q_i are momenta of particles and holes and are subject to 'Fermi sea' restriction $p_i > a \geqslant q_i$. On the other hand, in (64) $p = \mathrm{e}^{\mathrm{i}x}$ and $q = \mathrm{e}^{\mathrm{i}y}$ represent complex coordinates of particles and holes and are not subject to this restriction. Due to the restriction on the values of p_i and q_i the τ-function (63) has only positive Fourier components in all odd flows.

The simplest 1-phase and 2-phase solutions are written below as a function of space $t_1 = \mathrm{i}x$ and time $t_2 = -\mathrm{i}t$.

- 1-phase solution describes one particle-hole excitation:

$$\tau_m = 1 + A\frac{f_m^+(p)f_m^-(q)}{p - q}\,\mathrm{e}^{\mathrm{i}(p-q)(x-vt)}. \tag{65}$$

The solution is characterized by two positive numbers p and $-q$ from $\mathbb{Z} + a$ labelling particle-hole momenta and an arbitrary m-independent constant A, which may depend on p, q. It propagates with velocity $v = p + q$.

- 2-phase solution (two particle-hole excitations):

$$\begin{aligned}\tau_m &= 1 + A_1\frac{f_m^+(p_1)f_m^-(q_1)}{p_1 - q_1}\,\mathrm{e}^{\mathrm{i}(p_1-q_1)x-\mathrm{i}(p_1^2-q_1^2)t} + A_2\frac{f_m^+(p_2)f_m^-(q_2)}{p_2 - q_2}\,\mathrm{e}^{\mathrm{i}(p_2-q_2)x-\mathrm{i}(p_2^2-q_2^2)t}\\ &\quad + A_1A_2\det_{i,j=1,2}\left(\frac{f_m^-(q_j)f_m^+(p_i)}{p_i - q_j}\right)\mathrm{e}^{\mathrm{i}(p_1-q_1)(x-v_1t)}\,\mathrm{e}^{\mathrm{i}(p_2-q_2)(x-v_2t)}.\end{aligned}$$

The multi-phase solutions in (64) allow for rational limit. Keeping $v_i = p_i + q_i$ constant while taking the limit $p_i - q_i \to 0$, one obtains rational solutions corresponding to N-solitons with v_i being asymptotic velocities of solitons. However, in (63) and its particular cases (65), (66) this limit does not exist. Indeed, under the condition $p_i > a \geqslant q_i$ the limit $p_i - q_i \to 0$ implies $p_i, q_i \to 0$ and therefore it is impossible to have finite v_i in this limit. We see that the matrix element (14) does not contain soliton solutions of the MKP-hierarchy.

4. Conclusion

We have shown that the real-time dynamics of certain matrix elements (14) of one-dimensional (non-relativistic) fermions obeys nonlinear integrable equations of the modified KP-hierarchy. This evolution is driven by a commutative set of Hermitian Hamiltonians (13)

$$H_k = \int :\psi^\dagger(x)\left(-\mathrm{i}\frac{\partial}{\partial x}\right)^k\psi(x):\frac{\mathrm{d}x}{2\pi}.$$

It has to be contrasted with a traditional representation of the τ-function of the modified KP-hierarchy (5) [1, 3] in terms of fermions. The latter describes an evolution driven by a commutative set of (non-Hermitian) current operators (4)

$$J_k = \int \mathrm{e}^{\mathrm{i}kx} :\psi^\dagger(x)\psi(x):\frac{\mathrm{d}x}{2\pi}.$$

In these two cases the flows have different meaning. The former are coordinates of physical spacetime, while the latter are deformation parameters of the coherent state.

The matrix element we studied in this paper is understood as an overlap of the evolved coherent fermionic state $\langle G(\mathbf{t})|$ with a 'probe' boundary state $|B_m\rangle$. This matrix element appears in variety of physical problems involving the phenomena referred to as the orthogonality catastrophe. Among them are problems related to the Fermi-edge singularity, quantum impurity and tunnelling in effectively one-dimensional systems.

The fact that questions of quantum dynamics in one dimension are linked to the τ-function, indicates that the kinetics is essentially nonlinear is subject to instabilities inherited from the nonlinear dynamics. In a forthcoming paper [14] we describe the Whitham approach to shock wave-type solutions for electronic systems.

An essential element of our proof of the MKP-hierarchy is the representation of a fermionic mode by a Bose field (44), (46) in the action on a boundary state,

$$\psi_p \sim f^+(p)\,\mathrm{e}^{-\Phi(p)}.$$

This is a representation in momentum space. It is valid in a subspace of the Fock space consisting of excitations on top of the boundary state.

Again it has to be contrasted with the traditional 'bosonization'—representation of Fermi field in terms of a Bose field with respect to the Fermi vacuum

$$\psi(x) \sim \mathrm{e}^{\varphi(x)}.$$

This is a representation in coordinate space. It is valid in a subspace of the Fock space consisting of excitations on top of the Fermi vacuum.

We emphasize that although the matrix elements (5) and (14) obey similar MKP equations they give essentially different solutions of those equations. For example, (14) gives only solutions satisfying analyticity requirements in odd flows. The latter condition excludes soliton solutions.

Finally, we emphasize that the Schur decomposition (61) of the τ-functions similar to (14) (an essential element of the proof) has appeared in [7] in the problem of counting of Hurwitz numbers and in [6] in constructions of τ-functions of hypergeometric type. Similar objects arise in supersymmetric gauge theories [7, 15] and also in matrix quantum mechanics.

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